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Galois structure of modular forms of even weight

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ABSTRACT

We calculate the equivariant Euler characteristics of powers of the canonical sheaf on certain modular curves over $\mathbb{Z}$ which have a tame action of a finite abelian group. As a consequence, we obtain information on the Galois module structure of modular forms of even weight having Fourier coefficients in certain ideals of rings of cyclotomic algebraic integers.

Contents

1. Introduction

This paper concerns the calculation of certain equivariant Euler characteristics which arise in studying the action of diamond Hecke operators on spaces of even weight modular forms. A recent method, developed by Chinburg et al. in [CPT], shows how to calculate the Euler characteristic of coherent sheaves on projective flat schemes over $\mathbb{Z}$ on which a finite abelian group acts tamely. Up to a class of order at most 2, their technique neglects no torsion information if the base scheme has dimension less than 5. Their method shows that the Euler characteristic differs by computable terms

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from certain classes in Grothendieck groups which have "n-cubic" structures. This idea was motivated by previous work of Pappas in [P] and [P1]. In the main theorem of [CPT], a precise formula for the Euler characteristic is given. Furthermore, the structure of the lattice of weight 2 cusp forms for $\Gamma_1(p)$ which have integral Fourier expansions as a module for the action of the finite group of diamond Hecke operators is determined as an application. This is done by calculating the equivariant Euler characteristic $\overline{\chi}^P(X, \mathcal{O}_X)$ where X is a certain integral model of the modular curve $X_1(p)$.

In this paper, we calculate the equivariant Euler characteristic of the k-th power of certain fibral twists of the canonical sheaf of an integral model of the modular curve $X_1(p)$. We use such twists in order to be better able to relate our results to Galois modules of modular forms. In the process, we find a lower bound for the degree of the twist which is sufficient to guarantee that the first cohomology group vanishes. Consequently, the structure of the lattice of cusp forms of weight $2k$ having a given Nebentypus character and Fourier coefficients in certain ideals of rings of algebraic integers can be obtained as a module for the diamond Hecke operators. We state our results precisely below.

We will use the same notation as [CPT]. Let $p \equiv 1 \bmod 24$ be a prime and let $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$. Suppose $\chi : \Gamma \to \mu_r \subset \mathbb{Z}[\zeta_r]^*$ is a character of prime order $r|(p-1)$ with $r > 3$. Let $\delta \geqslant 0$ be an integer. Define $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ to be the $\mathbb{Z}[\zeta_r]$-module of cusp forms of weight $2k$, level p and of Nebentypus character χ having Fourier coefficients at ∞ in $p^{-\delta k}\mathbb{Z}[\zeta_r]$. (The parameter δ specifies the multiple of a fibral divisor we will use to twist the dualizing sheaf on certain integral models of modular curves.) The locally free $\mathbb{Z}[\zeta_r]$-module $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ is of rank $n(\chi) = \frac{(2k-1)(p-25)}{12}$. For $a \in (\mathbb{Z}/r\mathbb{Z})^*$ let $\{a\}$ be the unique integer in the range $0 < \{a\} < r$ having residue class a, and let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ be the automorphism for which $\sigma_a(\zeta_r) = \zeta_r^{\{a\}}$. Define $\omega_r : (\mathbb{Z}/r\mathbb{Z})^* \to \mathbb{Z}_r^*$ to be the Teichmuller character. The ring $\mathbb{Z}$ (resp. $\mathbb{Z}_r$) is embedded into the profinite completion $\hat{\mathbb{Z}} = \prod_{l \text{ prime}} \mathbb{Z}_l$ of $\mathbb{Z}$ diagonally (resp. via the factor $l = r$). Then a modified quadratic Stickelberger element of $\hat{\mathbb{Z}}[\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})]$ can be defined by

$$\theta_2 = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^*} \frac{(p-1)}{24r^2}\big(\{a\}^2 - \omega_r(a)^2\big)\sigma_a^{-1}. \tag{1.1}$$

We also define the truncated Stickelberger element $[\theta_1]$ by

$$[\theta_1] = \sum_{0 < q \leqslant kr - 1 + [\frac{-2kr}{(p-1)}],\ (q,r)=1} m(k,r) q \sigma_q^{-1} \tag{1.2}$$

where $m(k,r)$ is an integer depending on k and r. The truncated sum-element $[\theta_0]$ can be also defined by

$$[\theta_0] = \sum_{0 < q \leqslant kr - 1 + [\frac{-2kr}{(p-1)}],\ (q,r)=1} n(k,\delta,r) \sigma_q^{-1} \tag{1.3}$$

where $n(k,\delta,r)$ is an integer depending on k, δ and r.

Since the ideal class group $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$ is finite, θ_2, $[\theta_1]$ and $[\theta_0]$ all act on this group. Let $\mathcal{P}_\chi$ be the prime ideal of $\mathbb{Z}[\zeta_r]$ over (p) with the property that the reduction of χ modulo $\mathcal{P}_\chi$ is the $\frac{p-1}{r}$-th power of the identity character $(\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{F}_p^*$.

Let X_1 be the integral model of the modular curve $X_1(p)$ constructed in Theorem 4.2 of [CPT]; some properties of X_1 are recalled in Theorem 2.1 of Section 2. The group $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ acts faithfully on X_1. Let H be a subgroup of Γ of index r, we let X_H to be the quotient X_1/H and $\mu : X_1 \to X_H$ the quotient map. The special fiber of X_1 over p has two irreducible components. The unramified component D^1_∞ is distinguished from the other component, D^1_0 by the fact that D^1_∞ intersects the cuspidal section ∞.

Theorem 1.1. *Suppose $\mathfrak{A} \subset \mathbb{Z}[\zeta_r]$ is an ideal with ideal class $\theta_2 \cdot [\mathcal{P}_{\chi_0}] - [\theta_1] \cdot [\mathcal{P}_{\chi_0}] - [\theta_0] \cdot [\mathcal{P}_{\chi_0}]$. Then we have*

$$\overline{\chi}^P\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big) = [\mathfrak{A}]$$

in $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$.

Theorem 1.2. *Assuming the same terms as in the preceding theorem, for $\delta > 2 + r$,*

$$S_{2k,\delta}\big(\Gamma_1(p), p^{-\delta k}\mathbb{Z}[\zeta_r]\big)_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{A}$$

as $\mathbb{Z}[\zeta_r]$-modules.

This extends the corresponding theorem of [CPT] to higher weight cusp forms.

2. Coarse moduli schemes X_1, X_0 and X_H

The following theorem is shown in [CPT, Theorem 4.2] and concerns some particular integral models of the modular curves. This theorem follows from the work of Deligne and Rapoport [DR], Katz and Mazur [KM] and Conrad et al. [CES].

Theorem 2.1.

a. *The scheme $X_H \to \mathrm{Spec}(\mathbb{Z})$ is a flat projective curve, X_H is normal Cohen–Macaulay and $X_H[1/p] \to \mathrm{Spec}(\mathbb{Z}[1/p])$ is smooth (where $X_H[1/p] = X_H \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$). The special fiber of X_H over p has two irreducible components D_∞^H and D_0^H distinguished by the fact that D_∞^H intersects the cuspidal section ∞; these have multiplicities 1 and $(p-1)/(2 \cdot \#H)$ respectively.*
b. *The scheme X_H has at most two non-regular points which are rational singularities and lie on $D_0^H - (D_0^H \cap D_\infty^H)$. Their exact number depends on $\#H \bmod 6$: In particular, if 6 divides $\#H$ then there are no such points and X_H is regular. In particular, when $H = \{1\}$ there are two non-regular points on X_1. There is a morphism $b : X_1' \to X_1$ which is a rational resolution of those two singular points and a morphism $c : X_1' \to \mathcal{X}_1$ which is a sequence of blow-downs of exceptional curves such that $\mathcal{X}_1$ is regular and all the geometric fibers of $\mathcal{X}_1 \to \mathrm{Spec}(\mathbb{Z})$ are integral. Let $U = X_1 - D_0^{\{1\}} \subset X_1$. Then $U \to \mathrm{Spec}(\mathbb{Z})$ is smooth, b and c are isomorphisms on $b^{-1}(U)$ and $\mathcal{X}_1 - c(b^{-1}(U))$ has dimension 0.*
c. *The special fiber of X_0 over p is reduced with simple normal crossings. Each of the two irreducible components $D_\infty = D_\infty^\Gamma$ and $D_0 = D_0^\Gamma$ are isomorphic to $\mathbb{P}^1_{\mathbb{F}_p}$. We have $D_0 \cdot D_\infty = g_0 + 1 = (p-1)/12$ where g_0 is the genus of X_0.*
d. *Assume that 6 divides the order $\#H$.*
   i. *The morphism $\pi_H : X_H \to X_0$ is a tame $G = \Gamma/H$ cover of regular projective curves and $\pi_H[1/p] : X_H[1/p] \to X_0[1/p]$ is a G-torsor.*
   ii. *The morphism π_H is totally ramified over the generic point of D_0, and unramified over the generic point of D_∞. The irreducible components D_0^H and D_∞^H of $X_H \otimes_{\mathbb{Z}} \mathbb{F}_p$ are the (reduced) inverse images of D_0 and D_∞ under π_H. The character χ_0 giving the action of G on the cotangent space of the codimension 1 generic point of D_0^H equals $\omega^{-2\cdot\#H}$, where $\omega : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{F}_p^*$ is the Teichmuller (identity) character.*

3. Lattices of cusp forms

If R is a subring of $\mathbb{C}$ we will denote by $S_{2k}(\Gamma_1(p), R)$ the R-module of cusp forms $F(z) = \sum_{n \geqslant 1} a_n e^{2\pi i n z}$ of weight $2k$ for the congruence subgroup $\Gamma_1(p) \subset \mathrm{PSL}_2(\mathbb{Z})$ whose Fourier coefficients a_n belong to R. (These are the Fourier coefficients "at the cusp ∞".) Define $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}R)_\chi$ to be the R-module of cusp forms of weight $2k$, level p and of Nebentypus character χ having Fourier coefficients at ∞ in $p^{-\delta k}R$. In particular, if $\delta = 0$, we have $S_{2k,0}(\Gamma_1(p), R) = S_{2k}(\Gamma_1(p), R)$. Recall from

Theorem 2.1 and the paragraph before the statement of Theorem 1.1 that X_1 is a particular integral model of the modular curve $X_1(p)$ and $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ acts faithfully on X_1. The special fiber of X_1 over p has two irreducible components. The unramified component D^1_∞ is distinguished from the other component, D^1_0 by the fact that D^1_∞ intersects the cuspidal section ∞.

Proposition 3.1. *There are Γ-equivariant isomorphisms*

$$S_{2k,\delta}\big(\Gamma_1(p), p^{-\delta k}\mathbb{Z}\big) \simeq \mathrm{H}^0\big(X_1, \omega^{\otimes k}_{X_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big) \tag{3.1}$$

where the Γ-action on $S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z})$ is via the diamond operators and $\omega_{X_1/\mathbb{Z}}(\delta D^1_\infty)$ denotes the twisted canonical (dualizing) sheaf of $X_1 \to \mathrm{Spec}(\mathbb{Z})$ along the divisor δD^1_∞.

Proof. We follow the proof of Proposition 4.5 in [CPT]. We take $G(q) = \sum_{n\geqslant 1}(a_n/p^{\delta k})q^n \in S_{2k,\delta}(\Gamma_1(p), p^{-\delta k}\mathbb{Z})$ with $q = e^{2\pi i z}$ and consider $G(q)(dq/q)^{\otimes k}$ as a regular differential over $\mathrm{Spec}(\mathbb{Z}[\![q]\!])$. Then, repeating the same argument in the proof, which depends on the Kodaira–Spencer map, we can easily obtain the Γ-equivariant isomorphism,

$$\Phi\colon S_{2k,\delta}\big(\Gamma_1(p), p^{-\delta k}\mathbb{Z}\big) = \mathrm{H}^0\big(\mathcal{X}_1, \omega^{\otimes k}_{\mathcal{X}_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big). \tag{3.2}$$

We still need to prove the following isomorphism,

$$\mathrm{H}^0\big(\mathcal{X}_1, \omega^{\otimes k}_{\mathcal{X}_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big) = \mathrm{H}^0\big(X_1, \omega^{\otimes k}_{X_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big). \tag{3.3}$$

By Theorem 2.1, we have morphisms $b: X'_1 \to X_1$ and $c: X'_1 \to \mathcal{X}_1$. By [CES, p. 380], b is a rational resolution of two singular points Q_E and Q_F which correspond to the points $j = 0$ and $j = 1728$ on the fiber of X_1 over p. Furthermore, c is a sequence of blow-downs of regular curves.

The desired equality (3.3) follows from Corollaries 3.3 and 3.5 below.

Lemma 3.2. *Let the morphism $b: X'_1 \to X_1$ be rational resolution of the singularities at the points Q_E and Q_F as in Theorem 2.1. Then we have the following equality $\omega_{X'_1} = b^*\omega_{X_1}$.*

Proof. This follows from Proposition 5.1 in [Ar]. $\square$

Corollary 3.3. *The morphism $b: X'_1 \to X_1$ gives an isomorphism*

$$\mathrm{H}^0\big(X'_1, \omega^{\otimes k}_{X'_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big) = \mathrm{H}^0\big(X_1, \omega^{\otimes k}_{X_1/\mathbb{Z}}\big(k\delta D^1_\infty\big)\big). \tag{3.4}$$

Proof. Using the projection formula, we get

$$R^i b_*\big(b^*\big(\omega^{\otimes k}_{X_1}\big(k\delta D^1_\infty\big)\big)\big) \simeq \omega^{\otimes k}_{X_1}\big(k\delta D^1_\infty\big) \otimes R^i b_*\mathcal{O}_{X'_1}. \tag{3.5}$$

By taking $i = 0$, we get

$$R^0 b_*\big(b^*\big(\omega^{\otimes k}_{X_1}\big(k\delta D^1_\infty\big)\big)\big) \simeq \omega^{\otimes k}_{X_1}\big(k\delta D^1_\infty\big) \tag{3.6}$$

because, $\mathcal{O}_{X_1} = b_*\mathcal{O}_{X'_1}$ which simply follows from the fact that X_1 is normal and b is birational as in Lemma 2.1 in [Ch].

Now, using the Leray spectral sequence,

$$\mathrm{H}^i\big(X_1, R^j b_*\big(b^*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)\big) \Rightarrow \mathrm{H}^{i+j}\big(X_1', b^*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big) \tag{3.7}$$

and by choosing $i = j = 0$ we get the desired result. □

Lemma 3.4. *Suppose $d : Y \to X$ is a blow-up of a regular point P on a normal projective curve X over $\mathbb{Z}$ and that D is a divisor on X which does not pass through P. Let E be the exceptional curve on Y with respect to d. Then*

$$\omega_Y^{\otimes k}(kd^*D) = d^*\big(\omega_X^{\otimes k}(kD)\big) \otimes \mathcal{O}_Y(kE) \tag{3.8}$$

for $k \geqslant 0$ and this isomorphism gives rise to an isomorphism

$$\mathrm{H}^0\big(Y, \omega_Y^{\otimes k}(kd^*D)\big) = \mathrm{H}^0\big(X, \omega_X^{\otimes k}(kD)\big). \tag{3.9}$$

Proof. Let $Y \xrightarrow{d} X$ be the blow-up of the surface X at a regular point P. We have an exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(kE) \to \mathcal{O}_{kE}(kE) \to 0. \tag{3.10}$$

After tensoring this sequence by $d^*(\omega_X^{\otimes k}(kD))$, we obtain the exact sequence

$$\begin{aligned} 0 &\to d^*\big(\omega_X^{\otimes k}(kD)\big) \otimes \mathcal{O}_Y \to d^*\big(\omega_X^{\otimes k}(kD)\big) \otimes \mathcal{O}_Y(kE) \\ &\to d^*\big(\omega_X^{\otimes k}(kD)\big) \otimes \mathcal{O}_{kE}(kE) \to 0. \end{aligned} \tag{3.11}$$

By Proposition 3.3, Chapter 5 in [Ha],

$$\omega_Y(d^*D) \simeq d^*\big(\omega_X(D)\big)(E) \tag{3.12}$$

we have

$$\omega_Y^{\otimes k}(kd^*D) \simeq d^*\big(\omega_X^{\otimes k}(kD)\big)(kE). \tag{3.13}$$

If we restrict this sheaf to kE we get the last term in the sequence (3.11)

$$\omega_Y^{\otimes k}(kd^*D)|_{kE} \simeq d^*\big(\omega_X^{\otimes k}(kD)\big)(kE)|_{kE}. \tag{3.14}$$

Let's show that $\mathrm{H}^0(Y, \omega_Y^{\otimes k}(kd^*D)|_{kE})$ is trivial. We have

$$0 \to \mathcal{I}^j/\mathcal{I}^{j+1} \to \mathcal{O}_Y/\mathcal{I}^{j+1} \to \mathcal{O}_Y/\mathcal{I}^j \to 0 \tag{3.15}$$

where $\mathcal{I}$ is the ideal sheaf of E and $j = 0, 1, \ldots, k-1$. After twisting the sequence by $\omega_Y^{\otimes k}(kd^*D)$ we obtain

$$0 \to \mathcal{I}^j/\mathcal{I}^{j+1} \otimes \omega_Y^{\otimes k}(kd^*D) \to \mathcal{O}_Y/\mathcal{I}^{j+1} \otimes \omega_Y^{\otimes k}(kd^*D) \to \mathcal{O}_Y/\mathcal{I}^j \otimes \omega_Y^{\otimes k}(kd^*D) \to 0. \tag{3.16}$$

Since E is isomorphic to $\mathbb{P}^1$, a line bundle on E has non-trivial global sections if and only if its degree is non-negative. Using K_Y for the divisor of ω_Y in the adjunction formula,

$$(K_Y + E) \cdot E = -2 \tag{3.17}$$

gives

$$K_Y \cdot E = -1 \tag{3.18}$$

the degree of the first sheaf in this sequence is

$$\big(k(K_Y + D) - jE\big) \cdot E = -k + j \tag{3.19}$$

since

$$D \cdot E = 0, \qquad E \cdot E = -1. \tag{3.20}$$

Therefore,

$$\mathrm{H}^0\big(Y, \mathcal{O}_Y/\mathcal{I}^{j+1} \otimes \omega_Y^{\otimes k}(kd^*D)\big) \hookrightarrow \mathrm{H}^0\big(Y, \mathcal{O}_Y/\mathcal{I}^{j} \otimes \omega_Y^{\otimes k}(kd^*D)\big) \tag{3.21}$$

for all j. Since the right hand cohomology group in (3.21) is clearly 0 when $j = 0$, we conclude that $H^0(Y, d^*(w_X^{\otimes k}(kD)_{kE}))$ is trivial.

This shows, $\mathrm{H}^0(Y, \omega_Y^{\otimes k}(kd^*D)|_{kE})$ is trivial when we take $j = 0$.

Taking the cohomology of the sequence (3.11) gives a long exact cohomology sequence

$$0 \to \mathrm{H}^0\big(Y, d^*\big(\omega_X^{\otimes k}(kD)\big)\big) \to \mathrm{H}^0\big(Y, \omega_Y^{\otimes k}(kd^*D)\big) \to \mathrm{H}^0\big(Y, \omega_Y^{\otimes k}(kd^*D)|_{kE}\big) \to \cdots. \tag{3.22}$$

We conclude $\mathrm{H}^0(Y, d^*(\omega_Y^{\otimes k}(kD))) \simeq \mathrm{H}^0(Y, \omega_Y^{\otimes k}(kd^*D))$. Now the only thing that we need to show is that $\mathrm{H}^0(Y, d^*(\omega_X^{\otimes k}(kD))) \simeq \mathrm{H}^0(X, \omega_X^{\otimes k}(kD))$.

We use the projection formula,

$$R^i d_*\big(d^*\big(\omega_X^{\otimes k}(kD)\big)\big) \simeq \omega_X^{\otimes k}(kD) \otimes R^i d_* \mathcal{O}_Y. \tag{3.23}$$

A standard argument from Hartshorne [Ha, p. 387, Prop. 3.4] gives $R^i d_* \mathcal{O}_Y = 0$ if $i > 0$, and $R^0 d_* \mathcal{O}_Y = \mathcal{O}_X$ for $i = 0$. Now, using the Leray spectral sequence,

$$\mathrm{H}^i\big(X, R^j d_*\big(d^*\big(\omega_X^{\otimes k}(kD)\big)\big)\big) \Rightarrow \mathrm{H}^{i+j}\big(Y, d^*\big(\omega_X^{\otimes k}(kD)\big)\big) \tag{3.24}$$

and by choosing $i = j = 0$ we get the desired result. $\square$

Corollary 3.5. *The morphism $c : X_1' \to \mathcal{X}_1$ gives an isomorphism*

$$\mathrm{H}^0\big(X_1', \omega_{X_1'/\mathbb{Z}}^{\otimes k}\big(k\delta D_\infty^1\big)\big) \simeq \mathrm{H}^0\big(\mathcal{X}_1, \omega_{\mathcal{X}_1/\mathbb{Z}}^{\otimes k}\big(k\delta D_\infty^1\big)\big). \tag{3.25}$$

4. Galois structure of modular forms

Let's start this section by defining the module of the "twisted" modular forms of weight $2k$ on X_H as follows:

$$S_{2k,\delta}\big(\Gamma_H(p), p^{-\delta k}\mathbb{Z}\big) := S_{2k,\delta}\big(\Gamma_1(p), p^{-\delta k}\mathbb{Z}\big)^H.$$

We will try to calculate it here. Recall that μ is the quotient morphism $X_1 \to X_H$. The Leray spectral sequence gives

$$\mathrm{H}^i\big(X_H, R^j \mu_*\big(\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)\big) \Rightarrow \mathrm{H}^{i+j}\big(X_1, \omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big) \tag{4.1}$$

setting $i=0$, $j=0$, we get:

$$\mathrm{H}^0\big(X_H,\mu_*\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\simeq \mathrm{H}^0\big(X_1,\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big). \tag{4.2}$$

Therefore by Proposition 3.1

$$S_{2k,\delta}\big(\Gamma_H(p),p^{-\delta k}\mathbb{Z}\big)=\mathrm{H}^0\big(X_1,\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)^H \tag{4.3}$$

and

$$\begin{aligned}\mathrm{H}^0\big(X_1,\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)^H &\simeq \mathrm{H}^0\big(X_H,\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\\ &\simeq \mathrm{H}^0\big(X_H,\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big).\end{aligned} \tag{4.4}$$

Since $\mu^*(\omega_{X_H}(\delta D_\infty^H))$ and $\omega_{X_1}(\delta D_\infty^1)$ are line bundles, one of them can be written as a twist of the other. This twist is supported along the ramification locus since the line bundles are isomorphic on the complement of the ramification locus. Now we can write

$$\omega_{X_1}\big(\delta D_\infty^1\big)\simeq \mu^*\big(\omega_{X_H}\big(\delta D_\infty^H\big)\big)\big(R^1\big) \tag{4.5}$$

where R^1 is supported on the ramification locus. Also, if $p\equiv 1 \bmod 24$ then the ramification locus of the map horizontal divisors $j=0$ and $j=1728$. Their ramification degrees are $\frac{p-1}{2r}$, 2 and 3 respectively. A local calculation as in [Ma, p. 74], shows that $R^1=(\frac{p-1}{2r})D_0^1+\overline{\{j=0\}}^1+2\overline{\{j=1728\}}^1$ where $\overline{\{j=0\}}^1$ and $\overline{\{j=1728\}}^1$ are closures of the generic points of each of the lines $j=0$ and $j=1728$ on X_1. When we take the k-th powers of the sheaves above we get

$$\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\simeq \mu^*\big(\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\big)\big(kR^1\big). \tag{4.6}$$

After taking the H-invariants, we obtain

$$\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)^H\simeq \big(\mu^*\big(\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\big)\big(kR^1\big)\big)^H. \tag{4.7}$$

We know that

$$\big(\mu^*\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\big)^H:=\big(\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\otimes_{O_{X_H}}\mathcal{O}_{X_1}\big)^H, \tag{4.8}$$

$$\big(\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\otimes_{O_{X_H}}\mathcal{O}_{X_1}\big)^H=\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\otimes_{O_{X_H}}\mathcal{O}_{X_1}^H=\omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big). \tag{4.9}$$

Therefore,

$$\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\simeq \omega_{X_H}^{\otimes k}\big(k\delta D_\infty^H\big)\otimes_{O_{X_H}}\big(\mu_*\mathcal{O}_{X_1}\big(kR^1\big)\big)^H. \tag{4.10}$$

Notice that $\mathcal{O}_{X_1}(kR^1)$ is allowing poles of at most order k times along $\overline{\{j=0\}}^1$, at most $2k$ along $\overline{\{j=1728\}}^1$ and $k(\frac{p-1-2r}{2r})$ along D_0^1. The group H acts on the sheaf $O_{X_1}(kR^1)$ and H stabilizes the divisor kR^1. Since the action of H on X_1 is tame, $\mu_*O_{X_1}(kR^1)$ is a locally free rank one $O_{X_H}[H]$ module. This implies that $(\mu_*O_{X_1}(kR^1))^H$ is a line bundle on O_{X_H}. We can identify this line bundle by viewing it as a subsheaf of the function field of X_H. By considering the valuations of local sections of $(\mu_* O_{X_1}(kR^1))^H$ at codimension one points of X_H, we find that

$$\bigl(\mu_*\mathcal{O}_{X_1}\bigl(kR^1\bigr)\bigr)^H = \mathcal{O}_{X_H}\left(\left[\frac{k(p-1-2r)}{(p-1)}\right]D_0^H + \left[\frac{k}{2}\right]\overline{\{j=0\}}^H + \left[\frac{2k}{3}\right]\overline{\{j=1728\}}^H\right) \tag{4.11}$$

where $[t]$ means that the rational number t is rounded to the nearest integer less than t. Thus,

$$\begin{aligned}&\bigl(\mu_*\bigl(\omega_{X_1}^{\otimes k}\bigl(k\delta D_\infty^1\bigr)\bigr)\bigr)^H\\ &\quad\simeq \omega_{X_H}^{\otimes k}\left(k\delta D_\infty^H + \left[\frac{k(p-1-2r)}{(p-1)}\right]D_0^H + \left[\frac{k}{2}\right]\overline{\{j=0\}}^H + \left[\frac{2k}{3}\right]\overline{\{j=1728\}}^H\right).\end{aligned} \tag{4.12}$$

Now, we will state a key proposition that allows us to associate Galois structure invariants in ideal class groups of rings of integers to lattices of modular forms. This is based on results of [Ri] (see [CPT]).

Proposition 4.1. *Let $\chi : \Gamma \to \mathbb{Z}[\zeta_r]^*$ be a 1-dimensional character of prime order $r \geqslant 5$ with kernel H. Let $G = \Gamma/H$ and suppose M is a finitely generated torsion-free $\mathbb{Z}[G]$-module. Define M^χ to be the $\mathbb{Z}[\zeta_r]$-module $(M \otimes \mathbb{Z}[\zeta_r]\chi^{-1})^G$.*

a. *There is a unique homomorphism $e'_\chi : \mathrm{G}_0(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ such that for all M as above, either $M^\chi = \{0\}$ and $e'_\chi([M]) = 0$ or M^χ is isomorphic to $\mathbb{Z}[\zeta_r]^s \oplus \mathfrak{U}$ for some integer $s \geqslant 0$ and a $\mathbb{Z}[\zeta_r]$-ideal $\mathfrak{U}$ in the ideal class $e'_\chi([M])$.*
b. *There is a unique isomorphism $t_\chi : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathbb{Z} \oplus \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ such that $t_\chi([P]) = (\mathrm{rank}_{\mathbb{Z}[G]}(P), e_\chi(\overline{[P]}))$ if P is a projective $\mathbb{Z}[G]$-module, where $\overline{[P]}$ is the image of P in $\mathrm{Cl}(\mathbb{Z}[G])$ and $e_\chi : \mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[\zeta_r])$ is the unique homomorphism such that $e_\chi(\overline{[P]}) = e'_\chi(f([P]))$ for all projective P, where $f : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{G}_0(\mathbb{Z}[G])$ is the forgetful homomorphism.*

5. Proof of the main theorem

We now compute the image of $\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ under the injective homomorphism $\Theta = \Theta_3 : \mathrm{Cl}(\mathbb{Z}[G]) \to C_{\mathbb{Z}}(G, 3)$, which is defined in [CPT], by applying their main result and using the isomorphism (4.11). This result allows us to calculate the equivariant Euler characteristic of a sheaf if there is a tame cover and if the sheaf can be written as a pullback from the quotient. In our case, let $\pi_H : X_H \to X_0$ be our cover. Since the index of H in Γ is the prime $r \geqslant 5$, the order of H is divisible by 6. By Theorem 2.1 π_H is ramified only at the fiber over p. By (4.11) we get the sheaf $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H$ in terms of $\omega_{X_H}^{\otimes k}(k\delta D_\infty^H)$ twisted by a certain divisor. Also, $\omega_{X_H}^{\otimes k}(k\delta D_\infty^H)$ can be written as a pullback of the sheaf $\omega_{X_0}^{\otimes k}(k\delta D_\infty)$ with some twist. Therefore, $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H$ can be written as the same pullback of the sheaf $\omega_{X_0}^{\otimes k}(k\delta D_\infty)$ with some twist. Let's try to see the relation between them.

Making local calculations similar to those in [Ma, p. 74], we can say

$$\omega_{X_H}^{\otimes k}\bigl(k\delta D_\infty^H\bigr) = \pi_H^*\omega_{X_0}^{\otimes k}\bigl(k\delta D_\infty^H + k(r-1)D_0^H\bigr).$$

And hence,

$$\bigl(\mu_*\bigl(\omega_{X_1}^{\otimes k}\bigl(k\delta D_\infty^1\bigr)\bigr)\bigr)^H \simeq \pi_H^*\omega_{X_0}^{\otimes k}\bigl(k\delta D_\infty^H + R^H + \eta D_0^H\bigr). \tag{5.1}$$

Here, we denote by R^H the divisor $[\frac{k}{2}]\overline{\{j=0\}}^H + [\frac{2k}{3}]\overline{\{j=1728\}}^H$ on X_H, which can be identified as a pullback of the divisor $R^0 = [\frac{k}{2}]\overline{\{j=0\}}^0 + [\frac{2k}{3}]\overline{\{j=1728\}}^0$ on X_0. We also note $\eta = k(r-1) + [\frac{k(p-1-2r)}{(p-1)}]$.

Considering the fundamental sequence

$$0 \to \mathcal{O}_{X_H}\left(-\eta D_0^H\right) \to \mathcal{O}_{X_H} \to \mathcal{O}_{\eta D_0^H} \to 0 \tag{5.2}$$

and tensoring by $\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H)$ we get

$$\begin{aligned} 0 &\to \pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H\right) \\ &\to \pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H + \eta D_0^H\right) \to \pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H + \eta D_0^H\right)\big|_{\eta D_0^H} \to 0. \end{aligned} \tag{5.3}$$

Since $(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H \simeq \pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H)$, then our sequence becomes

$$0 \to \pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^1 + R^H\right) \to \left(\mu_*\left(\omega_{X_1}^{\otimes k}\left(k\delta D_\infty^1\right)\right)\right)^H \to \mathcal{C} \to 0 \tag{5.4}$$

where $\mathcal{C} = \pi_H^*(\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H + \eta D_0^H))|_{\eta D_0^H}$ is also supported on D_0^H.

The last sequence implies the following relation between equivariant Euler characteristics,

$$\overline{\chi}^P\left(X_H, \left(\mu_*\left(\omega_{X_1}^{\otimes k}\left(k\delta D_\infty^1\right)\right)\right)^H\right) = \overline{\chi}^P\left(X_H, \pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H\right)\right) + \overline{\chi}^P(X_H, \mathcal{C}). \tag{5.5}$$

On the right hand side, the first equivariant Euler characteristic will be calculated easily using the main theorem of [CPT], and the second one will be calculated using the adjunction formula as follows.

We have the following exact sequence

$$0 \to I^{q-1}/I^q \to \mathcal{O}_{X_H}/I^q \to \mathcal{O}_{X_H}/I^{q-1} \to 0 \tag{5.6}$$

where I is the ideal sheaf of D_0^H. By induction on η and tensoring each sequence by $\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H)$ we get the following sum for the equivariant Euler characteristic of $\mathcal{C}$

$$\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1} \overline{\chi}^P\left(X_H, \left[\pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H\right) \otimes \left(I^q/I^{q+1}\right)\right]\big|_{D_0^H}\right). \tag{5.7}$$

Let us denote by $N_{D_0^H}^\vee$ the conormal bundle of D_0^H. Then (5.7) gives

$$\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1} \overline{\chi}^P\left(X_H, \left[\pi_H^*\omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H\right) \otimes \left({N_{D_0^H}^\vee}^{\otimes q}\right)\right]\big|_{D_0^H}\right). \tag{5.8}$$

Since $r|p-1$ we can identify 1-dimensional $\mathbf{Q}_p^*$-valued characters of G with modular $(\mathbb{Z}/p\mathbb{Z})^*$-valued characters. The character χ_0, defined in Theorem 2.1, is such a character. We denote by $[\chi_0]$ the $\mathbb{Z}[G]$-module of order p whose modular character is given by χ_0. This module is of finite projective dimension. Hence, it gives a class in $\mathrm{Cl}(\mathbb{Z}[G])$ which we will denote also by $[\chi_0]$.

Let us revisit the calculation of the Euler characteristic $\overline{\chi}^P(X_H, \mathcal{C})$. The group G acts by part d(ii) of Theorem 2.1, on the cotangent space of codimension 1 generic point of D_0^H, also it acts transitively on the sections of $N_{D_0^H}^\vee$. So, the class of the Euler characteristic $\overline{\chi}^P(X_H, \mathcal{C})$ can be identified by finding the numerical Euler characteristic of the sheaf $[\pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H)\otimes (N_{D_0^H}^{\vee}{}^{\otimes q})]|_{D_0^H}$ on D_0^H for any q, which can be calculated by using Riemann–Roch. We obtain the following formula for the numerical Euler characteristic:

$$\begin{aligned}&\chi\big(X_H, \big[\pi_H^*\omega_{X_0}^{\otimes k}\big(k\delta D_\infty^1 + R^H\big)\otimes \big(N_{D_0^H}^{\vee}{}^{\otimes q}\big)\big]\big|_{D_0^H}\big)\\ &\quad = \deg\big(\pi_H^*\omega_{X_0}^{\otimes k}\big(k\delta D_\infty^H + R^H\big)\otimes \big(N_{D_0^H}^{\vee}{}^{\otimes q}\big)\big|_{D_0^H}\big) + 1 - g(X_H)\end{aligned} \tag{5.9}$$

where the degree of the sheaf in the formula above can be calculated by adjunction formula. We get

$$\begin{aligned}&\deg\big(\pi_H^*\omega_{X_0}^{\otimes k}\big(k\delta D_\infty^H + R^H\big)\otimes \big(N_{D_0^H}^{\vee}{}^{\otimes q}\big)\big|_{D_0^H}\big)\\ &\quad = -\frac{kr(p-1)}{12} + 2k - \frac{k(r-1)(p-1)}{12r}\\ &\qquad + \frac{kr\delta(p-1)}{12} + \left(\left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right]\right)r - q\eta\frac{(p-1)}{12}.\end{aligned} \tag{5.10}$$

So, we can write the Euler characteristic as

$$\overline{\chi}^P(X_H, \mathcal{C}) = \sum_{q=0}^{\eta-1}\big(m(k,r)q + n(k,\delta,r)\big)\big[\chi_0^q\big] \tag{5.11}$$

for integers

$$m(k,r) = -\eta\frac{(p-1)}{12} \tag{5.12}$$

and

$$n(k,\delta,r) = -\frac{kr(p-1)}{12} + 2k - \frac{k(r-1)(p-1)}{12r} + \frac{kr\delta(p-1)}{12} + \left(\left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right]\right)r \tag{5.13}$$

depending on k, δ and r.

Now let's turn to the calculation of $\overline{\chi}^P(X_H, \pi_H^*\omega_{X_0}^{\otimes k}(k\delta D_\infty^H + R^H))$. We will use the main result of [CPT] for this calculation, it is better to keep the same notation as in [CPT] for the convenience of the reader.

The field $\mathbf{Q}_p$ already contains a primitive $(p-1)$-st root of unity. Hence, we may take $R'_{(p)} = \mathbb{Z}_p$. We find the image of the Euler characteristic under the injective map, Θ in the category of cubic structure $C_{\mathbb{Z}}(G;3)$ as in [CPT]. The image is given by the idéle $(b_v)_v \in \mathbf{A}^*_{f,\mathbf{Q}[G^3]}$ which is 1 at all places $v \neq (p)$ and is such that

$$(\chi\otimes\phi\otimes\psi)(b_{(p)}) = p^{-T((\chi-1)(\phi-1)(\psi-1))} \tag{5.14}$$

with $T: \mathrm{Ch}(G)_p \to \mathbf{Q}$ the function associated to the cover $X_H \otimes_{\mathbb{Z}} \mathbb{Z}_p \to X_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in the main theorem. For $a \in \mathbb{Z}/r\mathbb{Z}$ let $\{a\}$ be the unique integer in the range $0 \leqslant \{a\} < r$ having residue class a. Using Eq. (3.15) in [CPT], T becomes

$$T(\psi)=\frac{p-1}{12}\cdot\left(-\frac{\{a\}_r^2}{2r^2}-\bigl(1-2k(1+\delta)\bigr)\frac{\{a\}_r}{2r}\right)+\bigl(1-2k(1+\delta)\bigr)\frac{\{a\}_r}{r} \tag{5.15}$$

where $\psi=\chi_0^{-a}$, $\chi_0=\omega^{\frac{(p-1)}{r}}$ and $\omega:(\mathbb{Z}/p\mathbb{Z})^*\to\mathbb{Z}_p^*$ is the Teichmuller character.

For $a\in\mathbb{Z}/r\mathbb{Z}$ define $\omega_r(a)=0$ if $a=0$, and otherwise let $\omega_r(a)\in\mathbb{Z}_r\subset\hat{\mathbb{Z}}$ be the Teichmuller character associated to r.

Define

$$T_1(\psi)=-\frac{p-1}{12}\left(\frac{\omega_r(a)^2}{2r^2}\right)\quad\text{and}\quad T_2(\psi)=\bigl(1-2k(1+\delta)\bigr)\frac{\{a\}_r}{r} \tag{5.16}$$

where $\psi=\chi_0^{-a}$ as above. We extend $\psi\to T_i(\psi)$ to a function on the character ring $\mathrm{Ch}(G)_p$ by additivity. Since $p\equiv 1 \bmod 24$ and $r|\frac{p-1}{24}$, we can define $\beta=(\beta_v)_v$ with $\beta_v\in\hat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Z}_v[G]^*$ by

$$\psi(\beta_v)=\begin{cases}1, & \text{if } v\neq(p);\\ p^{-T(\psi)+T_1(\psi)+T_2(\psi)}, & \text{if } v=(p).\end{cases} \tag{5.17}$$

Since $\mathrm{Cl}(\mathbb{Z}[G])$ is a torsion group, β defines a unique class $[\beta]$ in $\mathrm{Cl}(\mathbb{Z}[G])$.

Following the same argument as in [CPT, p. 31], it can be shown that

$$\overline{\chi}^P\bigl(X_H,\pi_H^*\omega_{X_0}^{\otimes k}\bigl(k\delta D_\infty^H+R^H\bigr)\bigr)=[\beta]. \tag{5.18}$$

To complete the proof of our theorem, we will find $-e_\chi(\overline{\chi}^P(X_H,(\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H))$ after calculating $-e_\chi(\overline{\chi}^P(X_H,\mathcal{C}))$ and adding with our result (5.18). By choosing a suitable element of $\Delta=\mathrm{Gal}(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ to apply, we can reduce to the case in which

$$\chi=\chi_0=\omega^{\frac{(p-1)}{r}}. \tag{5.19}$$

The character χ induces a homomorphism

$$e_\chi:\mathrm{Cl}\bigl(\mathbb{Z}[G]\bigr)\to\mathrm{Cl}\bigl(\mathbf{Q}(\zeta_r)\bigr),\qquad [P]\to\bigl[\bigl(P\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_r]\chi^{-1}\bigr)^G\bigr] \tag{5.20}$$

where the second pair of brackets denotes the "Steinitz ideal class" of the corresponding torsion free $\mathbb{Z}[\zeta_r]$-module. For $a\in(\mathbb{Z}/r\mathbb{Z})^*$ let $\sigma_a\in\mathrm{Gal}(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ be the automorphism for which $\sigma_a(\zeta_r)=\zeta_r^a$. We will denote by the same symbol the automorphism of the ideal class group $\mathrm{Cl}(\mathbf{Q}(\zeta_r))$ given by $\mathcal{A}\mapsto\sigma_a(\mathcal{A})$. Since $p\equiv 1 \bmod r$ the prime ideal (p) splits completely in $\mathbf{Q}(\zeta_r)$. We can identify complex-valued and $\mathbf{Q}_p$-valued characters of $G=\mathbb{Z}/r\mathbb{Z}$ by choosing a primitive $(p-1)$-st root of unity in $\mathbb{Z}_p$. We make this choice to that the composition $G\xrightarrow{\chi}\mathbb{Z}[\zeta_r]\to\mathbb{Z}_p/p\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ is equal to $\chi_0=\omega^{2|H|}$. Let $\mathcal{P}_\chi$ be the kernel of $\mathbb{Z}[\zeta_r]\to\mathbb{Z}/p\mathbb{Z}$, so that $\mathcal{P}_\chi$ is a prime ideal over p. With these conventions from the definition of e_χ in Proposition 4.1 we have $-e_\chi([\chi_0])=[\mathcal{P}_{\chi_0}]$ and hence

$$-e_\chi\bigl(\bigl[\chi_0^a\bigr]\bigr)=\bigl[\sigma_a^{-1}(\mathcal{P}_{\chi_0})\bigr],\quad\text{if } (a,r)=1,\qquad e_\chi\bigl(\bigl[\chi_0^a\bigr]\bigr)=0,\quad\text{otherwise.} \tag{5.21}$$

So, when we apply $-e_\chi$ to $\overline{\chi}^P(X_H,\mathcal{C})$ we get

$$\begin{aligned}-e_\chi\bigl(\overline{\chi}^P(X_H,\mathcal{C})\bigr)&=\sum_{q=0}^{\eta-1}\bigl(m(k,r)q+n(k,\delta,r)\bigr)\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}]\\&=\sum_{q=0}^{\eta-1}m(k,r)q\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}]+\sum_{q=0}^{\eta-1}n(k,\delta,r)\sigma_q^{-1}\cdot[\mathcal{P}_{\chi_0}].\end{aligned} \tag{5.22}$$

Recalling the definitions of each of the terms used, we get the following equality,

$$-e_\chi\big(\overline{\chi}^P\big(X_H,\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)\big)$$
$$=\theta_2\cdot[\mathcal{P}_{\chi_0}]\frac{(1-p)(2k-1)}{24r}-\theta_1\cdot[\mathcal{P}_{\chi_0}]-[\theta_1]\cdot[\mathcal{P}_{\chi_0}]-[\theta_0]\cdot[\mathcal{P}_{\chi_0}] \tag{5.23}$$

where

$$\theta_1=\sum_{a\in(\mathbb{Z}/r)^*}\{a\}\sigma_a^{-1}\in\mathbb{Z}[\Delta], \tag{5.24}$$

$$[\theta_1]=\sum_{0<q\leqslant kr-1+[\frac{-2kr}{(p-1)}],\,(q,r)=1} m(k,r)q\sigma_q^{-1} \tag{5.25}$$

and

$$[\theta_0]=\sum_{0<q\leqslant kr-1+[\frac{-2kr}{(p-1)}],\,(q,r)=1} n(k,\delta,r)\sigma_q^{-1}. \tag{5.26}$$

By Stickelberger's Theorem, θ_1 annihilates $\mathrm{Cl}(\mathbb{Z}[\zeta_r])$, so the proof is complete.

5.1. A lower bound for δ

In this part we try to find a lower bound on δ that allows us to calculate the Galois structure of the lattice of twisted cusp forms explicitly. In our main calculation, we calculated the equivariant Euler characteristic for any value of δ, which is namely,

$$\overline{\chi}^P\big(X_H,\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)=\big[H^0\big(X_H,\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)\big)\big]$$
$$-\big[H^1\big(X_H,\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)\big] \tag{5.27}$$

in $\mathrm{Cl}[\mathbb{Z}(G)]$. If we arrange δ so that the first cohomology group vanishes, then we obtain a precise formula for the twisted cusp forms. The details are given in the following corollary.

Corollary 5.1. *There exists δ_0 such that for every $\delta>\delta_0$, we have the following: Suppose $\mathfrak{A}\subset\mathbb{Z}[\zeta_r]$ is an ideal with ideal class $\theta_2\cdot[\mathcal{P}_{\chi_0}]-[\theta_1]\cdot[\mathcal{P}_{\chi_0}]-[\theta_0]\cdot[\mathcal{P}_{\chi_0}]$. Then,*

$$S_{2k,\delta}\big(\Gamma_1(p),\mathbb{Z}[\zeta_r]\big)_\chi\simeq\mathbb{Z}[\zeta_r]^{n(\chi)-1}\oplus\mathfrak{A} \tag{5.28}$$

as $\mathbb{Z}[\zeta_r]$-modules.

Proof. With the notations of the theorem, recall that $S_{2k,\delta}(\Gamma_1(p),p^{-\delta k}\mathbb{Z}[\zeta_r])_\chi$ is the $\mathbb{Z}[\zeta_r]$-submodule of $S_{2k,\delta}(\Gamma_1(p),p^{-\delta k}\mathbb{Z}[\zeta_r])$ consisting of twisted cusp forms of weight $2k$ and of Nebentypus character χ whose n-th Fourier coefficients at ∞ are in the form of $\frac{r}{p^{\delta k}}$ where r in $\mathbb{Z}[\zeta_r]$. Proposition 3.1 and its proof together with the fact that formation of the canonical sheaf commutes with the base change $\mathbb{Z}\to\mathbb{Z}[\zeta_r]$ imply that $S_{2k,\delta}(\Gamma_1(p),\mathbb{Z}[\zeta_r])\simeq S_{2k,\delta}(\Gamma_1(p),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_r]$. Propositions 3.1 and 4.1 now give an isomorphism of (torsion free) $\mathbb{Z}[\zeta_r]$-modules

$$S_{2k,\delta}\big(\Gamma_1(p),p^{-\delta k}\mathbb{Z}[\zeta_r]\big)_\chi\simeq\mathrm{H}^0\big(X_H,\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)^\chi.$$

The projective class $\chi^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H) \in \mathrm{K}_0(\mathbb{Z}[G])$ has the property that

$$\begin{aligned} & f\big(\chi^P\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)\big) \\ &\quad = \big[\mathrm{H}^0\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)\big] - \big[\mathrm{H}^1\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)\big] \end{aligned} \tag{5.29}$$

where $f : \mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{G}_0(\mathbb{Z}[G])$ is the forgetful homomorphism. If P is a projective $\mathbb{Z}[G]$-module, then $\mathbf{Q} \otimes_{\mathbb{Z}} P$ is a free $\mathbf{Q}[G]$-module, so $\mathrm{rank}_{\mathbb{Z}[G]}(P) = \mathrm{rank}_{\mathbb{Z}}(P)/r$. Therefore, using Riemann–Roch we get

$$\begin{aligned} & \mathrm{rank}_{\mathbb{Z}[\zeta_r]}\mathrm{H}^0\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big) - \mathrm{rank}_{\mathbb{Z}[\zeta_r]}\mathrm{H}^1\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big) \\ &\quad = \frac{(2k-1)(g(X_H)-1)}{r} + \left[\frac{k(p-1-2r)}{(p-1)}\right] + \left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right] \end{aligned} \tag{5.30}$$

where $g(X_H)$ is the genus of X_H.

Because the generic fiber of $X_H \to X_0$ is étale of degree r, by the Hurwitz Theorem, we can say $(g(X_H) - 1)/r = g(X_0) - 1$ and we know that $g(X_0) = \frac{(p-13)}{12}$ hence,

$$n(\chi) = \frac{(2k-1)(p-25)}{12} + \left[\frac{k(p-1-2r)}{(p-1)}\right] + \left[\frac{k}{2}\right] + \left[\frac{2k}{3}\right]. \tag{5.31}$$

Let $\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ be the image of $\chi^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ in $\mathrm{Cl}(\mathbb{Z}[G])$. If we prove that $\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ vanishes when $\delta > \delta_0$ for some δ_0, we can easily conclude from (5.29) and (4.1) that there is an isomorphism of $\mathbb{Z}[\zeta_r]$-modules

$$S_{2k,\delta}\big(\Gamma_1(p), \mathbb{Z}[\zeta_r]\big)_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{U} \tag{5.32}$$

where $\mathfrak{U}$ is a $\mathbb{Z}[\zeta_r]$-ideal having ideal class $-e_\chi(\overline{\chi}^P(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H))$.

The only thing left to prove is the statement of the following lemma.

Lemma 5.2. $\mathrm{H}^1(X_H, (\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)$ *is trivial when* $\delta > \delta_0$ *for some* δ_0.

Proof. If we can show that $\mathrm{H}^1(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is torsion free (which is a necessary condition for duality), then the result will follow by duality as follows:

$$\mathrm{H}^1\big(X_H, \big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big) = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{H}^0\big(X_H, \big(\big(\mu_*\big(\omega_{X_1}^{\otimes k}\big(k\delta D_\infty^1\big)\big)\big)^H\big)^\vee\big), \mathbb{Z}\big). \tag{5.33}$$

Here $\mathrm{H}^0(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is trivial because of the degree of the sheaf is negative.

To show that $\mathrm{H}^1(X_H, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)^\vee)$ is torsion free it is enough to check that $\mathrm{H}^0((X_H)_\beta, ((\mu_*(\omega_{X_1}^{\otimes k}(k\delta D_\infty^1)))^H)_\beta^\vee)$ is trivial on each fiber β, this argument can be deduced from the proof of Theorem 12.11 [Ha]. If the fiber $\beta \neq p$, then it is just $\mathbb{P}^1$ and degree of the sheaf is negative implies result. Otherwise ($\beta = p$), we have two components namely D_0^H and D_∞^H, one of which is totally ramified and the other unramified. Let s be a global section of our sheaf, then its restriction to D_∞^H is zero since D_∞^H is reduced. Let's call W for the non-reduced component. So, $W^{red} = D_0^H$ and

$$O_W = O_{D_0^H} \oplus N \oplus N^{\otimes 2} \oplus \cdots \oplus N^{\otimes r-1} \tag{5.34}$$

where $N = O_{X_H}(-D_0^H)|_{D_0^H}$.

Recall the following equation,

$$\left(\mu_*\left(\omega_{X_1}^{\otimes k}\left(k\delta D_\infty^1\right)\right)\right)^H \simeq \pi_H^* \omega_{X_0}^{\otimes k}\left(k\delta D_\infty^H + R^H + \eta D_0^H\right). \tag{5.35}$$

Therefore, s is given by the r-tuple of sections s_i of the sheaf

$$N^{\otimes i}\left((1-k)K_H \cdot D_0^H - k\delta D_\infty^H \cdot D_0^H - \eta D_0^H \cdot D_0^H - R^H \cdot D_0^H\right). \tag{5.36}$$

We know that

$$N^{\otimes r} = \mathcal{O}_{X_H}\left(-rD_0^H\right)\big|_{D_0^H} = \mathcal{O}_{X_H}\left(D_\infty^H\right)\big|_{D_0^H} \tag{5.37}$$

then

$$\deg\left(N^{\otimes r}\right) = D_0^H \cdot D_\infty^H = \frac{(p-1)}{12} \tag{5.38}$$

which implies

$$\deg(N) = \frac{p-1}{12r}. \tag{5.39}$$

We also know that

$$K_H \cdot \left(D_\infty^H + rD_0^H\right) = 0 \quad \Rightarrow \quad K_H \cdot D_\infty^H = -rK_H \cdot D_0^H. \tag{5.40}$$

Adjunction formula gives

$$\left(K_H + D_\infty^H\right) \cdot D_\infty^H = 2g_{D_\infty^H} - 2 \tag{5.41}$$

and Hurwitz formula gives

$$2g_{D_\infty^H} - 2 = r(2g_{D_\infty} - 2) + \frac{(r-1)(p-1)}{12} \tag{5.42}$$

both together imply that

$$K_H \cdot D_\infty^H = -D_\infty^H \cdot D_\infty^H - 2r + \frac{(r-1)(p-1)}{12}. \tag{5.43}$$

Also,

$$D_\infty^H \cdot \left(D_\infty^H + rD_0^H\right) = 0 \quad \Rightarrow \quad D_\infty^H \cdot D_\infty^H = -rD_0^H \cdot D_\infty^H = \frac{-r(p-1)}{12}. \tag{5.44}$$

So,

$$K_H \cdot D_\infty^H = \frac{r(p-1)}{12} + \frac{(r-1)(p-1)}{12} - 2r \tag{5.45}$$

and

$$K_H \cdot D_0^H = -\frac{(p-1)}{12} - \frac{(r-1)(p-1)}{12r} + 2. \tag{5.46}$$

If we plug all these into the degree calculation of our sheaf we get

$$\begin{aligned}
&\frac{i(p-1)}{12r} - \frac{k\delta(p-1)}{12} + (1-k)\left(-\frac{(p-1)}{12} - \frac{(r-1)(p-1)}{12r} + 2\right) \\
&\quad + \left[\frac{k(p-1-2r)}{(p-1)}\right]\left(\frac{p-1}{12r}\right) - \left[\frac{k}{2}\right] - \left[\frac{2k}{3}\right] \\
&\quad \leqslant \frac{r(p-1)}{12} - \frac{k\delta(p-1)}{12} + (2k-1)\frac{(p-1)}{12} + \frac{(p-1)}{12r} + 2 - \frac{10k}{3}.
\end{aligned} \tag{5.47}$$

We want to find a lower bound to δ which makes the right hand side of this inequality negative. Thus we want

$$\frac{r(p-1)}{12} - \frac{k\delta(p-1)}{12} + (2k-1)\frac{(p-1)}{12} + \frac{(p-1)}{12r} + 2 - \frac{10k}{3} < 0. \tag{5.48}$$

This is equivalent to

$$\delta > 2 + \frac{1}{k}\left(r - 1 + \frac{1}{r} + \frac{24}{(p-1)}\right) - \frac{40}{(p-1)} \tag{5.49}$$

and remember that $r > 3$ and $24r$ divides $p-1$, therefore $r+2$ is going to be enough for δ_0. $\square$

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